Int. J. Solids Structures Vol. 21, No. 1, pp. 55–65, 1985
Printed in the U.S.A.

0020–7683/85 $3.00 + .00

ELASTIC-PLASTIC COMPOSITES AT FINITE STRAINS

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(*Received* 2 *August* 1984)

Abstract—For elastic-plastic composites at finite strains and rotations, Hill's self-consistent method is used to estimate the overall instantaneous elastic-plastic moduli. Results are applied to a porous metal whose matrix constitutive relations are characterized by a generalized J_2 plasticity model. As an illustration, uniaxial deformation of a porous elastic-plastic solid is considered, and conditions under which the overall uniform deformation becomes unstable by localized deformations, are examined.

1. INTRODUCTION

Following pioneering work of Voigt [1] and Reuss [2], models have been developed for estimating the overall moduli of elastic composites which undergo infinitesimal deformations; see, for example, MacKenzie [3], Hill [4, 5], Hershey [6], Kerner [7], Kröner [8], Hashin [9], Budiansky [10], Walpole [11], and more recent work of Nemat-Nasser, Iwakuma and Hejazi [12] and Iwakuma and Nemat-Nasser [13]. For elastic-plastic composites, Hutchinson [14] has used the self-consistent method of Hill [15] to estimate the effective instantaneous elastic-plastic moduli, again for infinitesimal deformations. Recently, Nemat-Nasser and Iwakuma [16] and Iwakuma and Nemat-Nasser [17] have formulated Hill's self-consistent method for finite strains and rotations, and have applied it to calculate the instantaneous moduli of polycrystalline solids.

In the present paper we use a phenomenological J_2 plasticity model to characterize the response of the matrix material in an elastic-plastic composite. We estimate the variation of some of the (macroscopic) parameters of this model by employing the numerical results of a microscopic calculation of the polycrystal plasticity of Iwakuma and Nemat-Nasser [17]. Then we outline the self-consistent formulation of the overall instantaneous moduli of a finitely deformed, two-phase elastic-plastic composite, and apply it to a porous metal consisting of an elastic-plastic matrix which contains randomly distributed, initially circular (in two dimensions) holes. In the numerical example, special attention is focused on the phenomenon of localized deformations which may emerge under overall uniform farfield stresses in such porous elastic-plastic materials.

2. MATRIX CONSTITUTIVE RELATIONS

The constitutive relations for the matrix are based on a modified J_2 flow potential and a yield function which approximates a vertexed yield condition by involving noncoaxiality between the Cauchy stress, σ_{ij}, and the plastic strain rate, d^p_{ij}, i.e.

$$d^{p\prime}_{ij} = \frac{1}{H}\frac{\sigma'_{ij}\,\sigma'_{kl}}{4\bar{\sigma}^2}\,\overset{\circ}{\sigma}_{kl} + A\left(\frac{\sigma'_{ij}}{\bar{\sigma}}\right)^{\circ}\bar{\sigma}, \tag{1}$$

where components with respect to a fixed rectangular Cartesian coordinate system are used, repeated indices are summed, prime denotes the deviatoric part, H is the hardening parameter, superposed $\circ$ denotes the Jaumann rate, $\overset{\circ}{\sigma}_{ij} = \dot{\sigma}_{ij} - w_{ik}\,\sigma_{kj} - w_{jk}\,\sigma_{ki}$ (w_{ij} is the spin tensor), $\bar{\sigma} = (\frac{1}{2}\,\sigma'_{ij}\,\sigma'_{ij})^{1/2}$, and A is the noncoaxiality parameter.

In eqn (1), the last term renders the plastic strain rate tensor noncoaxial with the Cauchy stress tensor. A quantity of this kind emerged in the double slip theory of Mandel [18] and later, de Josselin de Jong [19] and Spencer [20], in relation to the flow

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of granular materials. Later on, the same term was introduced by Rudnicki and Rice [21] to characterize the plasticity of rocks containing microfissures, and by Stören and Rice [22] to study localized deformation in metal sheets. Recently, Christoffersen, Mehrabadi and Nemat-Nasser [23] obtained the same noncoaxiality expression from a micromechanical modeling of granular materials that support applied loads through frictional contact. A similar quantity results in single-crystal constitutive relations; Asaro [24] and Nemat-Nasser, Mehrabadi and Iwakuma [25]. In a recent work, Nemat-Nasser [26] suggests calling the inverse of A the Mandel-Spencer noncoaxiality modulus. It can easily be verified that, without such noncoaxiality, the use of the J_2 flow potential and yield function with positive hardening precludes a prediction of localized deformation‡; Rudnicki and Rice [21], Stören and Rice [22], and Iwakuma and Nemat-Nasser [13].

Let the elastic part of the deformation rate be given by

$$d^e_{ij} = \frac{1}{2\mu}\,[\tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}]\{\overset{\circ}{\sigma}_{kl} + \sigma_{kl}\, d^e_{mm}\} \tag{2}$$

and set

$$\overset{\circ}{\sigma}_{ij} = \mathcal{L}_{ijkl} d_{kl}, \tag{3}$$

where d_{kl} is the deformation rate, d^e_{ij} is its elastic part,

$$\begin{aligned}\mathcal{L}_{ijkl} = {} & \hat{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \left(\frac{\mu}{1-2\nu} - \hat{\mu}\right)\delta_{ij}\delta_{kl} + (\mu - \hat{\mu})\frac{\sigma'_{ij}\sigma'_{kl}}{\bar{\sigma}^2} \\ & - \frac{2\mu}{1-2\nu}\frac{Y_{ij}\delta_{kl}}{1+Y_{mm}} - \frac{\mu^2}{h}\frac{\sigma'_{ij}}{\bar{\sigma}}\left\{(1+Y_{mm})\frac{\sigma'_{kl}}{\bar{\sigma}} - \frac{\sigma'_{mn}Y_{mn}}{(1-2\nu)\bar{\sigma}}\delta_{kl}\right\}\end{aligned} \tag{4}$$

and

$$\begin{aligned} & h = (1 + Y_{mm})(\mu + H), \\ & \frac{1}{2\hat{\mu}} = \frac{1}{2\mu} + A, \qquad Y_{ij} = \sigma_{ij}\Big/\left(\frac{2\mu}{1-2\nu} - \sigma_{kk}\right). \end{aligned} \tag{5}$$

If the Jaumann rate of the Cauchy stress instead of the Kirchhoff stress is used in eqn (2), then the corresponding moduli, $\mathcal{L}_{ijkl}$, are obtained from eqn (4) by simply setting $Y_{ij} \equiv 0$.

In terms of the nominal stress rate, eqn (3) yields

$$\frac{\dot{n}_{ij}}{\tau^0_y} = F_{ijkl}\frac{v_{k,l}}{\gamma_0}, \tag{6}$$

where τ^0_y is the yield stress in simple shear, $v_{k,l}$ is the velocity gradient (comma followed by an index denotes partial differentiation with respect to the corresponding coordinate) and

$$\begin{aligned} F_{ijkl} = {} & \frac{\mathcal{L}_{ijkl}}{\mu} + \dot{\gamma}_0\{S_{ij}\delta_{kl} - \tfrac{1}{2}S_{mj}(\delta_{ki}\delta_{lm} + \delta_{mk}\delta_{li}) \\ & + \tfrac{1}{2}S_{mi}(\delta_{jk}\delta_{ml} - \delta_{mk}\delta_{jl})\}, \quad S_{ij} = \frac{\sigma_{ij}}{\tau^0_y}. \end{aligned} \tag{7}$$

In order to examine the consistency of this model with the results from the microscopic calculations of Iwakuma and Nemat-Nasser [17], consider a uniaxial tensile test

‡ The statement applies to rigid-plastic materials. For elastic-plastic materials, localization is predicted at stresses of the order of the elastic shear modulus.

(in two dimensions) where except for the S_{11} all components of S_{ij} are zero. Then eqns (4) and (7) yield, for the nonzero components of F_{ijkl},

$$
\begin{aligned}
F_{1111} &= \frac{2(1-\nu)}{(1-2\nu)} - \frac{\mu}{h} - Y_{11}\left\{\frac{2}{(1-2\nu)(1+Y_{11})} - \frac{2\nu}{1-2\nu}\frac{\mu}{h}\right\},\\
F_{2222} &= \frac{2(1-\nu)}{(1-2\nu)} - \frac{\mu}{h} - Y_{11}\frac{2(1-\nu)}{(1-2\nu)}\frac{\mu}{h},\\
F_{1122} &= \frac{2\nu}{(1-2\nu)} + \frac{\mu}{h} - Y_{11}\left\{\frac{2}{(1-2\nu)(1+Y_{11})} - \frac{2(1-\nu)}{(1-2\nu)}\frac{\mu}{h}\right\} + \gamma_0 S_{11},\\
F_{2211} &= \frac{2\nu}{(1-2\nu)} + \frac{\mu}{h} - Y_{11}\frac{2\nu}{(1-2\nu)}\frac{\mu}{h},\\
F_{1212} &= F_{2112} = F_{2121} = \frac{\hat{\mu}}{\mu} - \frac{1}{2}\gamma_0 S_{11},\\
F_{1221} &= \frac{\hat{\mu}}{\mu} + \frac{1}{2}\gamma_0 S_{11},
\end{aligned}
\tag{8}
$$

where

$$
\frac{\mu}{h} = (1+Y_{11})^{-1}\frac{\mu}{h_0}, \quad \frac{\mu}{h_0} = \left(1+\frac{H}{\mu}\right)^{-1}, \quad Y_{11} = \frac{\gamma_0 S_{11}}{\frac{2}{1-2\nu} - \gamma_0 S_{11}}. \tag{9}
$$

Hence the shearing components of F_{ijkl} depend on $\hat{\mu}$ and thus on A, while the other components are affected by the parameter h. In particular, when the influence of the residual stresses is neglected by setting equal to zero the coefficient of γ_0, eqns (7) and (9) give

$$
\begin{aligned}
F_{1111} &= F_{2222} = \frac{2(1-\nu)}{(1-2\nu)} - \frac{\mu}{h_0},\\
F_{1122} &= F_{2211} = \frac{2}{(1-2\nu)} - F_{1111},\\
F_{1212} &= F_{1221} = F_{2112} = F_{2121} = \frac{\hat{\mu}}{\mu}.
\end{aligned}
\tag{10}
$$

From the definition of H and A in eqn (1) and from eqns (5) and (9), initially we must have

$$
\frac{\mu}{h_0} = 0 \quad \text{and} \quad \frac{\hat{\mu}}{\mu} = 1 \tag{11}
$$

in the elastic range. Similarly, for large plastic deformations, we have the limiting values,

$$
\frac{\mu}{h_0} \to 1 \quad \text{and} \quad \frac{\hat{\mu}}{\mu} \to 0, \tag{12}
$$

which, with eqn (10), lead to the numerical estimates of Iwakuma and Nemat-Nasser [17]. Therefore, the numerical results of these authors may be used to identify the two material parameters, μ/h_0 and $\hat{\mu}/\mu$.

Let α_1 and α_2 be non-negative parameters defined by

$$\frac{\mu}{h_0} = 1 - \alpha_1, \qquad \frac{\hat{\mu}}{\mu} = \alpha_2. \tag{13}$$

In order to estimate the functional form of α_1 and α_2, consider a simpler case where $Y_{11} = 0$. Then eqn (8) yields

$$\begin{aligned} F_{1111} &= F_{2222} = \frac{1}{1-2\nu} + \alpha_1, \\ F_{1122} &= \frac{1}{1-2\nu} - \alpha_1 + \gamma_0 S_{11}, \\ F_{2211} &= \frac{1}{1-2\nu} - \alpha_1, \\ F_{1221} &= \alpha_2 + \frac{1}{2}\gamma_0 S_{11}, \\ F_{1212} &= F_{2112} = F_{2121} = \alpha_2 - \frac{1}{2}\gamma_0 S_{11}. \end{aligned} \tag{14}$$

The necessary condition for inception of localized deformation in the present case is [see eqn (54)]

$$\gamma_0^2 \left(\frac{1}{1-2\nu} + \alpha_1\right)^2 S_{11}^2 = \frac{16}{1-2\nu}\alpha_1 \left(\frac{1}{1-2\nu} + \alpha_2\right)(\alpha_2 - \alpha_1), \tag{15}$$

which implies that

$$\alpha_2 \geq \alpha_1; \tag{16}$$

otherwise no localization is anticipated for any physically acceptable γ_0 and ν. Thus, as the axial strain is increased, the parameter μ/h_0 must approach one at a rate faster than the rate at which $(\hat{\mu}/\mu)$ is decaying. Such a tendency is in accord with the numerical results of Hutchinson [14] and Iwakuma and Nemat-Nasser [17].

Subject to eqn (16) and when eqn (15) holds, the orientation of localized deformation is given by [see eqn (53)]

$$\left(\frac{\nu_1}{\nu_2}\right)^2 = \frac{\dfrac{1}{1-2\nu}(\alpha_2 - \alpha_1) - \alpha_1\left(\dfrac{1}{1-2\nu} + \alpha_2\right)}{\left(\dfrac{1}{1-2\nu} + \alpha_1\right)\left(\alpha_2 + \dfrac{1}{2}\gamma_0 S_{11}\right)}. \tag{17}$$

For real values of ν_1/ν_2, the right-hand side must be positive, resulting in another necessary condition for localization,

$$\alpha_2 \geq 2\alpha_1/[1 - (1 - 2\nu)\alpha_1]. \tag{18}$$

Based on the results of Iwakuma and Nemat-Nasser [17], for the numerical calculation of composites the following explicit forms are used for α_1 and α_2:

$$\alpha_1 \equiv 0, \quad \alpha_2 \equiv 0 \quad \text{for} \quad \bar{\sigma}/\tau_Y^0 < 1, \tag{19}$$

$$\alpha_1 = \exp\left\{-\eta_1\left(\frac{\bar{\sigma}}{\tau_Y^0} - 1\right)^{1/2}\right\}, \quad \alpha_2 = \left\{\eta_2\left(\frac{\bar{\sigma}}{\tau_Y^0} - 1\right)^2 + 1\right\}^{-1} \quad \text{for} \quad \frac{\bar{\sigma}}{\tau_Y^0} \geq 1, \tag{20}$$

where the values of the positive parameters η_1 and η_2 are estimated using these authors' numerical results.

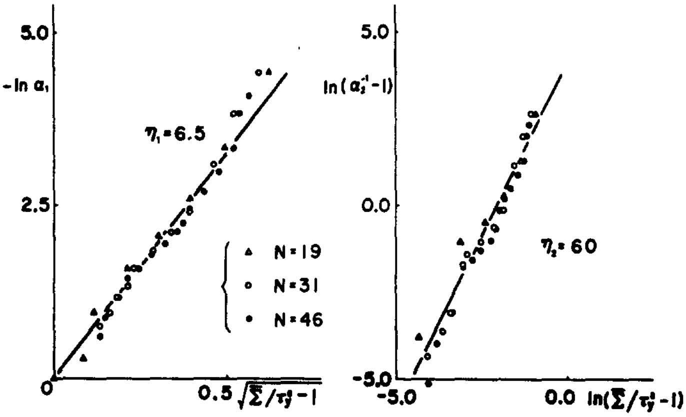

Fig. 1. Relations between global moduli and stress invariants; $\gamma_0 = 0$, no hardening.

To this end, we have plotted in Fig. 1, α_1 and α_2 as functions of the overall stress invariant Σ, for uniaxial deformation of a polycrystal, using $N = 19$, 31 and 46 crystal orientations in each quadrant, a two-dimensional problem. The details for these results are given by Iwakuma and Nemat-Nasser [17] and will not be repeated here. The solid lines in Fig. 1 represent eqn $(20)_{1,2}$ with

$$\eta_1 = 6.5, \qquad \eta_2 = 60. \tag{21}$$

Although the correlation shown in Fig. 1 is quite good, it should be emphasized that the expressions (20) are suggested here for illustration only and that other forms for α_1 and α_2 may be used. These expressions relate to the specific microstructure used in [17] for the single crystal, and, in fact, they may change as one considers other microstructures.

3. INSTANTANEOUS MODULI OF TWO-PHASE COMPOSITES

Consider an elasto-plastic composite consisting of a set of inclusions (voids), randomly distributed within a matrix. Let D be the domain occupied by a typical sample of volume V, and denote by Ω the collection of all inclusions in D. Define the overall constitutive relations in terms of the overall nominal stress rate, $\dot{N}_{ij}$, and the overall velocity gradient, $V_{i,j}$, by

$$\dot{N}_{ij} = \mathcal{F}^C_{ijkl} V_{k,l}, \qquad i, j, k, l = 1, 2, 3. \tag{22}$$

The objective is to estimate $\mathcal{F}^C_{ijkl}$ in terms of the instantaneous moduli, $\mathcal{F}^I_{ijkl}$ and $\mathcal{F}^M_{ijkl}$, of the inclusions and the matrix, and the volume fraction of inclusions,

$$f = V_I/V, \tag{23}$$

where V_I is the total volume of the inclusions. The local instantaneous moduli depend on the current local stresses, but are assumed to be independent of the deformation rates. The local stresses and hence the local moduli are estimated by the self-consistent method. As has been remarked by Hill [15], for two-phase composites the roles of the phases are reversible, although the matrix is not treated as an ellipsoidal inclusion in the formulation. Therefore the average local matrix stresses can be estimated with the aid of a concentration tensor, in exactly the same manner as for the inclusions. In the sequel, a representative volume subjected on its boundaries either to uniform traction rates, or to a velocity field compatible with uniform overall velocity gradients, is considered. Then the overall nominal stress rate $\dot{N}_{ij}$ measured with respect to the current configuration as reference is given by

$$\dot{N}_{ij} = \langle \dot{n}_{ij} \rangle_D, \tag{24}$$

where $\dot{n}_{ij}$ is the "local" nominal stress rate, and $\langle\ldots\rangle_D$ denotes volume average over D. In a similar manner, the overall velocity gradient becomes

$$V_{i,j} = \langle v_{i,j}\rangle_D, \tag{25}$$

where $v_{i,j}$ is the "local" velocity gradient.

From eqn (24) one obtains

$$\mathcal{F}^C_{ijkl} = (1 - f)\mathcal{F}^M_{ijmn}\mathcal{A}^M_{mnkl} + f\mathcal{F}^I_{ijmn}\mathcal{A}^I_{mnkl}, \tag{26}$$

where $\mathcal{A}^M_{ijkl}$ and $\mathcal{A}^I_{ijkl}$ are the concentration tensors for the matrix and the inclusion, respectively,

$$v^M_{i,j} = \mathcal{A}^M_{ijkl}\, V_{k,l}, \qquad v^I_{i,j} = \mathcal{A}^I_{ijkl}\, V_{k,l}. \tag{27}$$

Furthermore, eqn (25) leads to

$$(1 - f)\mathcal{A}^M_{ijkl} + f\, \mathcal{A}^I_{ijkl} = I_{ijkl}, \tag{28}$$

the identity tensor. From eqns (26) and (28), it follows that

$$\mathcal{F}^C_{ijkl} = \mathcal{F}^M_{ijkl} + f(\mathcal{F}^I_{ijmn} - \mathcal{F}^M_{ijmn})\mathcal{A}^I_{mnkl}. \tag{29}$$

When the inclusions are replaced by voids, one obtains

$$\mathcal{F}^C_{ijkl} = \mathcal{F}^M_{ijmn}\,\{I_{mnkl} - f\, \mathcal{A}^I_{mnkl}\}. \tag{30}$$

It now remains to define the constitutive relations for the matrix and calculate the concentration tensors.

It should be pointed out that only the nominal stress rate, $\dot{n}_{ij}$, and the velocity gradient, $v_{i,j}$, have the simple averaging properties defined by eqns (24) and (25), respectively, as has been discussed by Hill [27]; see also Havner [28] and Nemat-Nasser [26]. Other overall stress rates must therefore be *defined* in terms of $\dot{N}_{ij}$. For example if σ_{ij} is the *local* Cauchy stress, we have

$$\dot{\sigma}_{ij} = \dot{n}_{ij} + v_{ik}\,\sigma_{kj} - v_{k,k}\,\sigma_{ij}. \tag{31}$$

With the current configuration as the reference one, we also have

$$\sigma_{ij} = n_{ij} \quad \text{and} \quad \Sigma_{ij} = N_{ij}, \tag{32}$$

where Σ_{ij} is the overall Cauchy stress tensor; indeed, when the current configuration is used as the reference configuration, all commonly used stress measures equal the Cauchy stress tensor. From eqn (31), it is clear that $\langle\dot{\sigma}_{ij}\rangle_D$ does not, in general, equal $\dot{N}_{ij} + V_{ik}\,\Sigma_{kj} - V_{k,k}\,\Sigma_{ij}$. Therefore, we *define* $\dot{\Sigma}_{ij}$ by

$$\dot{\Sigma}_{ij} = \tfrac{1}{2}[\dot{N}_{ij} + \dot{N}_{ji} + V_{ik}\,\Sigma_{kj} + V_{jk}\,\Sigma_{ki}] - V_{k,k}\,\Sigma_{ij} \tag{33}$$

which also ensures the symmetry of $\dot{\Sigma}_{ij}$. The overall Cauchy stress may then be computed incrementally: given $\Sigma_{ij}(t)$ and $\dot{\Sigma}_{ij}(t)$ at time t, we have at time $t + \Delta t$,

$$\Sigma_{ij}(t + \Delta t) = \Sigma_{ij}(t) + \dot{\Sigma}_{ij}(t)\,\Delta t + O(\Delta t^2). \tag{34}$$

4. CONCENTRATION TENSOR

To calculate the local velocity gradient, $v_{i,j}$, in terms of the overall velocity gradient $V_{i,j}$ by means of the self-consistent method, one considers a homogeneous, infinitely extended solid of the overall instantaneous moduli $\mathcal{F}^C_{ijkl}$, containing an ellipsoidal inclusion Ω of instantaneous moduli $\mathcal{F}^I_{ijkl}$, and for the prescribed overall (constant) velocity gradient $V_{i,j}$ obtains the local $v_{i,j}$ in Ω. It has been shown by Iwakuma and Nemat-Nasser [17] that

$$v_{i,j} = \mathcal{A}^I_{ijkl} V_{k,l}, \tag{35}$$

where the concentration tensor $\mathcal{A}^I_{ijkl}$ depends on the geometry of Ω and on the overall moduli $\mathcal{F}^C_{ijkl}$ of the composite. It is defined by

$$\mathcal{A}^I_{ijkl} = [I_{ijkl} + J_{ijmn}(\mathcal{F}^C_{mnkl} - \mathcal{F}^I_{mnkl})]^{-1}, \tag{36}$$

with

$$J_{ijmn} = \int_\Omega \mathcal{G}_{in,mj}(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}', \tag{37}$$

where $\mathcal{G}_{in}(\mathbf{x} - \mathbf{x}')$ is Green's function that satisfies the following differential equations in an unbounded region:

$$\mathcal{F}^C_{mikl}\, \mathcal{G}_{kj,ml}(\mathbf{x} - \mathbf{x}') = -\delta_{ij}\, \delta(\mathbf{x} - \mathbf{x}'), \tag{38}$$

$\delta(\mathbf{x} - \mathbf{x}')$ being the Dirac delta function.

To show this, we observe that the nominal stress rate, $\dot{n}_{ij}$, is given by

$$\begin{aligned} \dot{n}_{ij} &= \mathcal{F}^C_{ijkl}\, v_{k,l} \quad \text{in} \quad D - \Omega \\ &= \mathcal{F}^I_{ijkl}\, v_{k,l} \quad \text{in} \quad \Omega \end{aligned} \tag{39}$$

and that we must have

$$v_{i,j} \to V_{i,j} \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{40}$$

Instead of eqns (39) and (40), we may consider

$$\dot{n}_{ij} = \dot{N}_{ij} + \dot{\hat{n}}_{ij}, \quad \dot{N}_{ij} = \mathcal{F}^C_{ijkl}\, V_{k,l} \quad \text{in} \quad D, \tag{41}$$

where the perturbation field $\dot{\hat{n}}_{ij}$ is expressed as

$$\dot{\hat{n}}_{ij} = \mathcal{F}^C_{ijkl}(\hat{v}_{k,l} - v^*_{k,l}), \qquad \hat{v}_{i,j} \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{42}$$

in which $v^*_{i,j}$ is the Eshelby-type "transformation" velocity gradient, identically zero outside of Ω. Consistency then requires that

$$\dot{n}_{ij} = \mathcal{F}^I_{ijkl}\, v_{k,l} \equiv \dot{N}_{ij} + \dot{\hat{n}}_{ij} = \mathcal{F}^C_{ijkl}(V_{k,l} + \hat{v}_{k,l} - v^*_{k,l}) \quad \text{in} \quad \Omega. \tag{43}$$

From equilibrium conditions $\dot{\hat{n}}_{ij,i} = 0$ and the definition of Green's function, eqn (38), it follows that

$$\hat{v}_{i,j} = -\mathcal{B}_{ijkl}\, v^*_{k,l} \quad \text{in} \quad \Omega, \tag{44}$$

where

$$\mathcal{B}_{ijkl} = -J_{ijmn}\, \mathcal{F}^C_{mnkl}, \tag{45}$$

with J_{ijmn} defined by eqn (37). By substituting into the consistency eqn (43), we immediately arrive at eqn (36).

Iwakuma and Nemat-Nasser [17] give details for calculating the integral in eqn (37).

In two dimensions, this can be done by means of simple contour integrations. Indeed, it can be shown in three dimensions that

$$J_{ijmn} = \frac{1}{2}\int_{-1}^{1} H_{ijmn}(\zeta_3)\,\mathrm{d}\zeta_3, \tag{46}$$

where

$$H_{jkmn}(\zeta_3) = \frac{-1}{2\pi i}\oint_{\gamma}\left[\frac{N_{nj}(\boldsymbol{\xi})\,\xi_k\,\xi_m}{z\,D(\boldsymbol{\xi})}\right]\mathrm{d}z, \tag{47}$$

in which $\xi_j = \zeta_j/a_j$ (no sum on j), $\zeta_1 = (1-\zeta_3^2)^{1/2}\cos\theta$ and $\zeta_2 = (1-\zeta_3^2)^{1/2}\sin\theta$, $z = e^{i\theta}$, $i = \sqrt{-1}$, γ is the unit circle for fixed $|\zeta_3| < 1$, a_j's are the principal radii of the ellipsoid Ω, and

$$N_{ij} = \text{cofactor of } \mathscr{F}^{C}_{kijl}\,\xi_k\,\xi_l,$$
$$D = \det|\mathscr{F}^{C}_{kijl}\,\xi_k\,\xi_l|, \quad |\boldsymbol{\xi}| = 1. \tag{48}$$

In two dimensions, eqn (46) becomes

$$J_{ijmn} = H_{ijmn}(\zeta_3 \to 0), \tag{49}$$

where the components of $\boldsymbol{\xi}$ now are $\left(\frac{1}{a_1}\cos\theta,\ \frac{1}{a_2}\sin\theta,\ 0\right)$.

The integral in eqn (37) can be evaluated as long as the operator $\mathscr{F}^{C}_{ijkl}\dfrac{\partial^2}{\partial x_j \partial x_l}$ is elliptic. This operator ceases to be elliptic when

$$\det|\mathscr{F}^{C}_{kijl}\,\nu_k\,\nu_l| = 0 \tag{50}$$

yields real roots for the unit vector $\boldsymbol{\nu}$. Then the material may not be able to support a uniform overall deformation, leading to localized deformations along planes normal to the direction $\boldsymbol{\nu}$. Apparently the condition in eqn (50) is identical with $D = 0$ in eqn (48). Thus there exists a relation between the condition for localization and the location of poles in the integral in eqn (47). For the two-dimensional case, eqn (50) reduces to

$$c_1\nu_1^4 - (c_2 + c_3)\nu_1^3\nu_2 - (c_4 + c_5)\nu_1^2\nu_2^2 + (c_6 + c_7)\nu_1\nu_2^3 + c_8\nu_2^4 = 0, \tag{51}$$

where the coefficients are given by

$$\begin{aligned}
c_1 &= F_1F_7 - F_3F_5, \quad c_6 = F_2F_{16} - F_4F_{14},\\
c_2 &= F_5F_{11} - F_7F_9, \quad c_8 = F_{10}F_{16} - F_{12}F_{14},\\
c_3 &= F_4F_5 + F_3(F_6 + F_{13}) - F_2F_7 - F_1(F_8 + F_{15}),\\
c_4 &= F_5F_{12} + F_{11}(F_6 + F_{13}) - F_7F_{10} - F_9(F_8 + F_{15}),\\
c_5 &= F_3F_{14} + F_4(F_6 + F_{13}) - F_1F_{16} - F_2(F_8 + F_{15}),\\
c_7 &= F_9F_{16} + F_{10}(F_8 + F_{15}) - F_{11}F_{14} - F_{12}(F_6 + F_{13}),
\end{aligned} \tag{52}$$

in which

$$\begin{aligned}
&F_1 = F_{1111}, \quad F_2 = F_{2111}, \quad F_3 = F_{1211}, \quad F_4 = F_{2211},\\
&F_5 = F_{1121}, \quad F_6 = F_{2121}, \quad F_7 = F_{1221}, \quad F_8 = F_{2221},\\
&F_9 = F_{1112}, \quad F_{10} = F_{2112}, \quad F_{11} = F_{1212}, \quad F_{12} = F_{2212},\\
&F_{13} = F_{1122}, \quad F_{14} = F_{2122}, \quad F_{15} = F_{1222}, \quad F_{16} = F_{2222},
\end{aligned} \tag{53}$$

where $F_{ijkl} = \mathscr{F}^{C}_{ijkl}/\mu$.

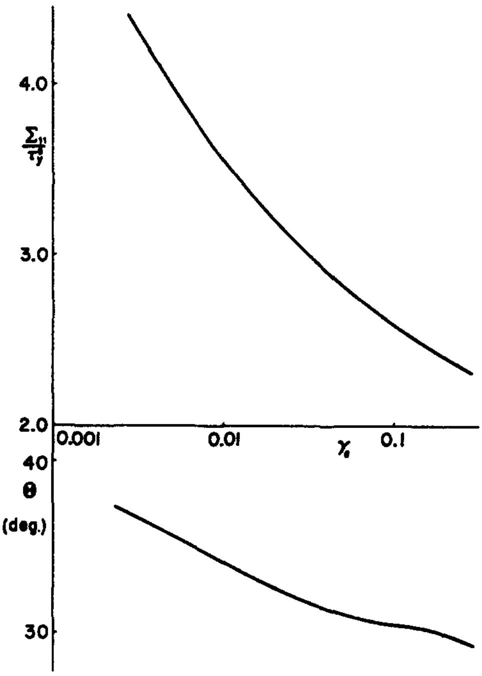

Fig. 2. Predictions of localized deformation for phenomenological model.

Equation (51) is a quartic in (ν_1/ν_2). For the elastic-plastic matrix material considered here, the moduli vary smoothly with deformation, and hence a double root for eqn (51) marks the (possible) incipience of localization. For uniaxial loading, symmetry reduces eqn (51) to

$$c_1\left(\frac{\nu_1}{\nu_2}\right)^4 - (c_4 + c_5)\left(\frac{\nu_1}{\nu_2}\right)^2 + c_8 = 0. \tag{54}$$

For the existence of a double root, we must have

$$(c_4 + c_5)^2 - 4\,c_1c_8 = 0 \tag{55}$$

which is a necessary condition for localization.

5. NUMERICAL RESULTS

Before considering a porous metal, we examine the inception of localized deformation in the uniaxial extension of a solid with constitutive relations (3)–(5), (20) and (21). Typical results for $\nu = \frac{1}{3}$ are presented in Fig. 2. Since the phenomenological constitutive relations assume elastic isotropy independently of the level of plastic flow, and ignore many other factors which the micromechanical modeling of polycrystals by Iwakuma and Nemat-Nasser [17] seems to include, their use inevitably leads to an overestimation of the critical stress.

Figure 3 presents uniaxial stress–strain curves of a porous metal with indicated void volume fractions. For the moduli, we have used expressions (4) and (7) with constitutive parameters fixed in the manner discussed in Section 2; Y_{ij} is set equal to zero. The curves indicated by $\gamma_0 = 0$ correspond to solutions in which the second order terms [i.e. terms proportional to γ_0 in eqn (7)] are neglected.

The calculations are done incrementally, where all quantities are referred to the current configuration, and then they are updated after each axial strain increment. To obtain the overall Cauchy stress, eqns (33) and (34) are used. The axial extension is defined by

$$\epsilon_1 = \Pi(1 + \partial\Delta U_1/\partial x_1) - 1, \tag{56}$$

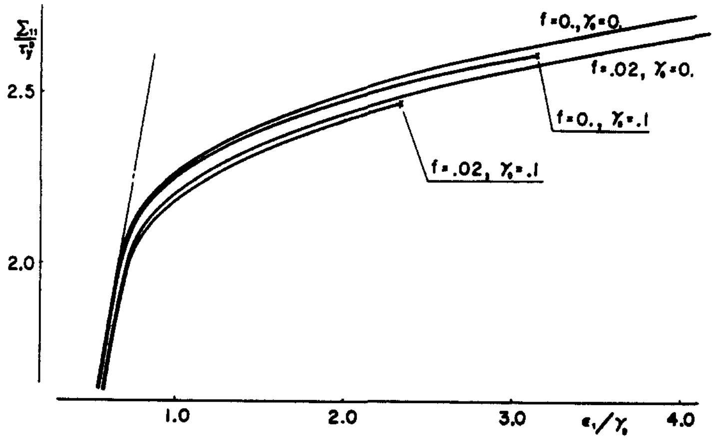

Fig. 3. Tensile stress–strain curves of porous solids.

where Π denotes the product taken over all preceding loading steps, and ΔU_1 is the displacement increment.

Let T_{ij} be the overall Kirchhoff stress. Its Jaumann rate then is

$$\overset{\circ}{T}_{ij} = \overset{\circ}{\Sigma}_{ij} + \Sigma_{ij} D_{kk} = \dot{\Sigma}_{ij} + \Sigma_{ij} D_{kk} - W_{ik}\Sigma_{kj} - W_{jk}\Sigma_{ki}. \tag{57}$$

Suppose now we wish to express the overall rate constitutive relations by

$$\overset{\circ}{T}_{ij} = L_{ijkl}\, D_{kl}, \tag{58}$$

where L_{ijkl} is the instantaneous modulus tensor. From eqns (33), (57) and (58) we obtain

$$L_{ijkl}\, D_{kl} = \tfrac{1}{2}\,(\mathcal{F}^C_{ijkl} + \mathcal{F}^C_{jikl} + \Sigma_{kj}\delta_{il} + \Sigma_{ki}\delta_{jl}) V_{k,l}. \tag{59}$$

The left-hand side of this equation is independent of the overall material spin. Hence, the right-hand side must satisfy

$$\begin{aligned} &\mathcal{F}^C_{ijkl} + \mathcal{F}^C_{jikl} + \Sigma_{kj}\delta_{il} + \Sigma_{ki}\delta_{jl} \\ &\quad = \mathcal{F}^C_{ijlk} + \mathcal{F}^C_{jilk} + \Sigma_{lj}\delta_{ik} + \Sigma_{li}\delta_{jk}. \end{aligned} \tag{60}$$

The modulus tensor L_{ijkl} is then given by

$$L_{ijkl} = \tfrac{1}{2}\,(\mathcal{F}^C_{ijkl} + \mathcal{F}^C_{jikl} + \Sigma_{kj}\delta_{il} + \Sigma_{ki}\delta_{jl}). \tag{61}$$

In general, L_{ijkl} is not symmetric with respect to the exchange of ij and kl.

Acknowledgment—This work has been supported by the Army Research Office under contract DAAG29-82-K-0147 to Northwestern University.

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